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PROOF OF THE NUMERICAL METHOD OF "DISCRETE VORTICES" FOR SOLVING SINGULAR INTEGRAL EQUATIONS

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I. K. LIFANOV and Ia. E. POLONSKII
(Moscow)

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Proof is given of the convergence of the numerical method of discrete vortices (see, e. g., [1-4]) in the solution of real one-dimensional singular integral equations of the first kind. It is shown that in the class of functions which are unlimited at one end of the integration segment and limited at the other, there exists a unique solution for which the Chaplygin - Joukowski condition is satisfied.

1. Statement of problem. Computation scheme. We consider the real one-dimensional singular integral equation (SIE) of the first kind

$$\int_a^b \gamma(x) \frac{K(x_0, x)}{x - x_0} dx = f(x_0) \tag{1.1}$$

with following conditions (conditions A): $f(x_0)$ satisfies Hölder's condition [5] with exponent $\alpha, H(\alpha)$, for $a < x_0 < b$; $H(x_0, x)$ satisfies the condition $H(\alpha)$ with respect to x_0 and x in the region $a \leqslant x_0, x \leqslant b$; $\gamma(x)$ is the unknown function to be determined in the class of functions which are limited for $x = b$ and unlimited for $x = a$. For $x = a$ function $\gamma(x)$ tends to infinity of order ν $(0 < \nu < 1)$ and can be, consequently, represented in the form $\gamma(x) = \varphi(x)(x - a)^{-\nu}$, where $\varphi(x)$ satisfies the condition $H(\alpha)$ for $a \leqslant x \leqslant b$.

The computation scheme of the considered method consists of dividing segment $[a, b]$ into n equal parts of length h on each of which at a distance of $^1/_4 h$ from their left-hand end are marked computation points x_i at which values of the sought function $\gamma(x_i)$ are calculated. At the same distance from the right-hand end are located check points x_{0j} $(i, j = 1, 2, \ldots, n)$ at which boundary conditions are satisfied. Thus each check point x_{0j} lies in the middle between adjacent computation points x_j and x_{j+1}, except point x_{0n} which divides segment $[x_n, b]$ in a 2 : 1 ratio.

The numerical method of discrete vortices consists of substituting for the SIE (1.1) of a system of n linear algebraic equations

$$\sum_{i=1}^{n} \gamma(x_i) \frac{K(x_{0j}, x_i)}{x_i - x_{0j}} h = f(x_{0j}), \quad j = 1, 2, \ldots, n \tag{1.2}$$

in values of the sought function $\gamma(x)$ determined at computation points x_i, with parameter x_0 assuming values x_{0j} $(i, j = 1, 2, \ldots, n)$.

It will be shown below that with increasing n solutions of system (1.1) approximate those of the SIE (1.1) at points x_i.

2. Existence and uniqueness of solution. Fulfilment of the Chaplygin-Joukowski condition. With the use of the regularization method [5] we represent SIE (1.1) in the form

$$\int_a^b \frac{\gamma(x)}{x - x_0} dx + \int_0^b k(x_0, x)\gamma(x)\, dx = \varphi(x_0) \tag{2.1}$$

$$k(x_0, x) = \frac{K(x_0, x) - K(x_0, x_0)}{K(x_0, x_0)}, \quad \varphi(x_0) = \frac{f(x_0)}{K(x_0, x_0)}$$

and assume that $K(x_0, x_0) \neq 0$ for $a \leqslant x_0 \leqslant b$ (equation of the normal kind).*

For the purpose of solution derivation Eq. (2.1), which for conditions A has a zero index, is equivalent to a Fredholm type of equation of the second kind

$$\gamma(x_0) - \frac{1}{\pi^2}\int_a^b N(x_0, x)\gamma(x)\, dx = -\frac{1}{\pi^2}\sqrt{\frac{b - x_0}{x_0 - a}}\, I_\varphi(x_0) \tag{2.2}$$

$$I_\varphi(x_0) = \int_a^b \sqrt{\frac{x - a}{b - x}} \frac{\varphi(x)}{x - x_0} dx$$

$$N(x_0, x) = \sqrt{\frac{b - x_0}{x_0 - a}} \int_a^b \sqrt{\frac{t - a}{b - t}} \frac{k(t, x)}{t - x_0} dt$$

For $x_0 = b$ we obtain from (2.2) that $N(b, x) = 0$ and $\gamma(b) = 0$. Hence the solution of SIE (1.1) which is limited for $x = b$, must necessarily vanish for $x = b$. In problems of aerodynamics mathematically defined by SIE (1.1) and conditions A (flow around a plate, a lattice of profiles, etc.) the relationship $\gamma(b) = 0$ means that the Chaplygin-Joukowski postulate is automatically satisfied in the derived solution.

If in the SIE (1.1) the kernel $K(x_0, x) \equiv 1$, we obtain the Cauchy SIE

$$\int_a^b \frac{\gamma(x)}{x - x_0} dx = f(x_0) \tag{2.3}$$

In this case we obtain from (2.1) and (2.2) that $k(x_0, x) \equiv 0$ and $N(x_0, x) \equiv 0$ and that the unique solution of SIE (1.1) is

$$\gamma(x_0) = -\frac{1}{\pi^2}\sqrt{\frac{b - x_0}{x_0 - a}}\, I_f(x_0) \tag{2.4}$$

The change of variable in (2.2) [5] yields a Fredholm integral equation of the second kind in which the kernel and the right-hand part are bounded functions. This shows that the solution of this equation exists and is unique. Hence Eq. (2.2) and Eq. (1.1), which is equivalent to it from the point of view of derivation of solution, have a unique solution.

3. Numerical solution of the Cauchy SIE. For the Cauchy SIE (2.3) we form the following system of n linear algebraic equations:

$$\sum_{i=1}^{n} \frac{\gamma(x_i)}{x_i - x_{0j}} h = f(x_{0j}), \quad j = 1, 2, \ldots, n \tag{3.1}$$

whose determinant is

$$\Delta^{(n)} = h^n \prod_{1 \leqslant m < p \leqslant n} (x_m - x_p)(-x_{0m} + x_{0p}) \Big/ \prod_{m,\, p=1}^{n} (x_m - x_{0p}) \tag{3.2}$$

which does not vanish.

We denote by $\Delta_i^{(n)}$ the determinant which is obtained from $\Delta^{(n)}$ by substituting in it the column of right-hand parts of system (3.1) for its i-th column, and by $\Delta_{i,j}^{(n)}$ the determinant which is obtained from $\Delta^{(n)}$ by deleting in it the j-th row and the i-th column $(i, j = 1, 2, \ldots, n)$.

From (3.1) and (3.2) we obtain

$$\gamma(x_i) = \frac{\Delta_i^{(n)}}{\Delta^{(n)}} = \frac{1}{\Delta^{(n)}} \sum_{j=1}^{n} (-1)^{i+j} \Delta_{i,j}^{(n)} f(x_{0j}) = \tag{3.3}$$

$$\sum_{j=1}^{n} (-1)^{j+i} \prod_{p=1}^{n} (x_i - x_{0p}) \prod_{m=1}^{n} (x_m - x_{0j}) \times$$

$$\left[\prod_{p=1}^{i-1} (x_p - x_i) \prod_{p=i+1}^{n} (x_i - x_p) \prod_{p=1}^{j-1} (-x_{0p} + x_{0j}) \prod_{p=j+1}^{n} (-x_{0j} + x_{0p}) \right]^{-1} \frac{f(x_{0j})}{h}$$

Using the singularity of the disposition of points x_i and x_{0j}, $(i, j = 1, 2, \ldots, n)$, we can transform formula (3.3) to

$$\gamma(x_i) = -\frac{1}{2^2} P_i^-(n) \sum_{j=1}^{n} P_j^+(n) \frac{f(x_{0j})}{x_{0j} - x_i} h \tag{3.4}$$

$$P_k^{\pm}(n) = \prod_{m=1}^{k-1} \left(1 \pm \frac{1}{2m}\right) \prod_{m=1}^{n-k} \left(1 \mp \frac{1}{2m}\right)$$

Using formula

$$\frac{(\beta+1)(\beta+2)\ldots(\beta+n)}{1 \cdot 2 \cdot \ldots \cdot n} = \frac{n^\beta}{\Gamma(\beta+1)} + O(n^{\beta-1})$$

known in the theory of gamma-functions [6], for $\beta = 1/2$ and $\beta = -1/2$ we obtain

$$P_i^-(n) = \frac{2}{\pi} \sqrt{\frac{n - i + 3/4}{i - 3/4}} + O([i(n-i+1)]^{-1/2}) + O((n-i+1)^{1/2} i^{-3/2}) \tag{3.5}$$

$$P_j^+(h) = \frac{2}{\pi} \sqrt{\frac{j - 1/4}{n - j + 1/4}} + O([j(n-j+1)]^{-1/2}) + O(j^{1/2}(n-j+1)^{-3/2})$$

The choice of points x_i and x_{0j} $(i, j = 1, 2, \ldots, n)$ implies that

$$\sqrt{\frac{n-i+3/4}{i-3/4}} = \sqrt{\frac{b - x_i}{x_i - a}}, \quad \sqrt{\frac{j-1/4}{n-j+1/4}} = \sqrt{\frac{x_{0j} - a}{b - x_{0j}}} \tag{3.6}$$

Hence formulas (3.3), (3.5) and (3.6) yield

$$\gamma(x_i) = -\frac{1}{\pi^2}\sqrt{\frac{b-x_i}{x_i-a}}\,S_f^i + \alpha(i, n-i+1), \tag{3.7}$$

$$S_f^i = \sum_{j=1}^{n}\sqrt{\frac{x_{0j}-a}{b-x_{0j}}}\,\frac{f(x_{0j})}{x_{0j}-x_i}\,h$$

where $\alpha(i, n-i+1)$ tend to zero with infinite increase of i and $n-i$ so that points x_i lie along segment $[a+\Delta, b-\delta]$, where Δ and δ are fixed arbitrary positive numbers. Comparing the derived values of $\gamma_e(x_i)$ in (3.7) with those of $\gamma_e(x_i)$ of the exact solution (2.4) of SIE (2.3) at point $x = x_i$ we find that to prove the convergence of $\gamma(x_i)$ to $\gamma_e(x_i)$ it is necessary to examine the convergence of the sum S_f^i to the value of integral $I_f(x_i)$, $i = 1, 2, \ldots, n$.

Since $f(x)$ satisfies condition $H(\alpha)$, and $\sqrt{1+x}$ the condition $H(^1/_2)$, we have

$$|\Delta I_{f(x_i)}| = |I_f(x_i) - S_f^i| \leqslant O\left[n^{-\alpha}\left(i-\frac{1}{2}\right)^{-1/2}(n-i+1)^{1/2}\ln n\right] + \tag{3.8}$$
$$O[n^{1/2}(n-i+1)^{-1}\ln n] + O\left[\left(n-i+\frac{1}{2}\right)^{-1/2}\left(i-\frac{1}{2}\right)^{1/2}\times\right.$$
$$\left.\left(\frac{1}{i-{}^1/_2} + \frac{1}{n-i+{}^1/_2}\right)\right] +$$
$$O\left[n^{1/2}\left(n-i+\frac{1}{2}\right)^{-1/2}\ln\left(1+\frac{2}{\sqrt{n-i+{}^1/_2}}\right)\right], \quad i = 1, 2, \ldots, n$$

Now, from (3.7) and (3.8) we obtain

$$|\gamma_e(x_i) - \gamma(x_i)| \leqslant \frac{1}{\pi^2}\sqrt{\frac{b-x_i}{x_i-a}}\,|\Delta I_f(x_i)| + \alpha(i, n-i+1) =$$
$$\beta(i, n-i+1)$$

where $\beta(i, n-i+1)$ tends to vanish with infinitely increasing i and $n-i+1$. Hence points x lie along segment $[a+\Delta, b-\delta]$. This proves the following theorem.

Theorem 1. The values of $\gamma(x_i)$ derived from the solution of system (3.1) for all points x_i lying alog segment $[a+\Delta, b-\delta]$, where Δ and δ are fixed arbitrary positive numbers, uniformly converges to the values obtained in the exact solution of the Cauchy SIE at the same points.

4. Numerical solution of the SIE of the first kind. Let us consider the SIE (1.1). We rewrite system (1.2) in the form

$$\sum_{i=1}^{n}\frac{\gamma(x_i)}{x_i-x_{0j}}\,h = \varphi(x_{0j}) - \sum_{p=1}^{n}k(x_{0j}, x_p)\,\gamma(x_p)\,h, \quad j = 1, 2, \ldots, n \tag{4.1}$$

where $k(x_{0j}, x_i)$ and $\varphi(x_{0j})$ are values of related functions defined in (2.1).

We solve this system for the unknown $\gamma(x_i)$, $i = 1, 2, \ldots, n$. This yields a formula similar to (3.4), where instead of $f(x_{0j})$ appears the right-hand part of formula (4.1).

Using formulas (3.5), (3.6) and (3.8), we obtain

$$\gamma(x_i) - \frac{1}{\pi^2}\int_a^b N(x_i, t)\,\gamma(t)\,dt = -\frac{1}{\pi^2}\sqrt{\frac{b-x_i}{x_i-a}}\,I_f(x_i) + \beta(i, n-i) \tag{4.2}$$

where $\beta(i, n-i) \to 0$ for all points $x_i \in [a+\Delta, b-\delta]$ when $n \to \infty$.

At point x_i the exact solution derived from (2.2) is

$$\gamma_f(x_i) - \frac{1}{\pi^2}\int_a^b N(x_i, t)\,\gamma_e(t)\,dt = -\frac{1}{\pi^2}\sqrt{\frac{b-x_i}{x_i-a}}\,I_f(x_i) \tag{4.3}$$

Comparison of (4.2) and (4.3) shows that Theorem 1 is valid for system (4.1) and the SIE (1.1).

5. Flow around a plate. Let us consider, as an example, the flow of a perfect incompressible fluid around a thin plate. We substitute a vortex sheet of intensity $\gamma(x)$ for the plate, and obtain for the latter SIE (2.3) in which $f(x_0) = -2\pi$ and $a = 0$, $b = 1$). Its exact solution is

$$\gamma_e(x) = 2\sqrt{(1-x)/x} \tag{5.1}$$

In conformity with the computation scheme of the method

$$x_i = h(i - 3/4), \quad x_0 = h(j - 1/4), \quad h = \frac{1}{n}; \quad i, \quad = 1, 2, \ldots, n. \tag{5.2}$$

Taking into account (5.2), we obtain for the system of linear equations of the numerical method corresponding to SIE (2.3) the following equation

$$\sum_{i=1}^{n} \frac{1}{j - i + 1/2}\,\gamma(x_i) = 2\pi, \quad j = 1, 2, \ldots, n. \tag{5.3}$$

Values of $\Delta_i = \gamma(x_i) - \gamma_e(x_i)$, where $\gamma(x_i)$ were obtained from the solution of system (5.3) and $\gamma_e(x_i)$ by formula (5.1) for $n = 20$, are tabulated below.

$x_i \cdot 10^4$	125	625	1125	1625	2125	2625	3125
$\Delta_i \cdot 10^2$	202.2	7.1	1.7	0.6	0.3	0.1	0.1
$x_i \cdot 10^4$	3625	4125	4625	5125	5625	6125	6625
$\Delta_i \cdot 10^2$	0	0	0	0	0	0	0
$x_i \cdot 10^4$	7125	7625	8125	8625	9125	9625	—
$\Delta_i \cdot 10^2$	−0.1	−0.1	−0.1	−0.1	−0.3	−0.9	—

It is seen that $\gamma_e(x_i)$ and $\gamma(x_i)$ diverge significantly only at the first point $x_1 = 0.0125$. This should have been expected, since $\gamma_e(0) = \infty$. At that point the relative error is 11.3%, at the last point x_{20} it is 2.3%, and at the remaining points it does not exceed 1%.

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DISPERSION PHENOMENA IN A BOILING BED

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V. L. GOLO and V. P. MIASNIKOV
(Moscow)
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Wave phenomena in a one-dimensional boiling bed are considered. A dispersion equation is derived which shows that instability in a boiling bed is weak in a fairly great number of cases. The Korteweg–de Vries–Burgers equation is obtained for waves of small but finite amplitude in the bed. Oscillations at the fronts of gas bubbles in a boiling bed are investigated. The linear increase of density fluctuations with distance from the bed bottom and the jump of fluctuation at the upper boundary are explained.

The mathematical analysis of stability of equations of a boiling bed appeared in several publications (see, e.g., [1 – 3]) in which it is shown that a strong instability exponentially increasing with time occurs in such beds. However no allowance was made in these for the boundedness of the bed in space, and the existence of homogeneous boiling beds at low fluidization rates is not explained.

Here the analysis of dispersion and the investigation of wave phenomena in a boiling bed is based on expansion in a small parameter introduced in [4].

The simple model of the boiling bed described in [1] is used for deriving the dispersion equation. The pseudo-gas viscosity and the pressure of pseudo-gas in particles are neglected, and the viscosity of the fluidizing gas is taken into account only in the interaction force between particles and gas. The model is one-dimensional, i. e. all functions depend only on the vertical component x.

In this case the input equations are of the form

$$\rho_s \varepsilon \left(\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x}\right) = -\varepsilon \frac{\partial p}{\partial x} - \rho_s \varepsilon G + \Phi$$

$$\rho_f (1-\varepsilon)\left(\frac{\partial v}{\partial t} + v \frac{\partial v}{\partial x}\right) = -(1-\varepsilon) \frac{\partial p}{\partial x} - \rho_f (1-\varepsilon) G - \Phi$$

$$\frac{\partial \varepsilon}{\partial t} + \frac{\partial \varepsilon u}{\partial x} = 0, \qquad \frac{\partial \varepsilon}{\partial t} - \frac{\partial}{\partial x}[(1-\varepsilon) v] = 0$$

$$\Phi = \frac{9}{2} \frac{\nu}{a^2} \rho_f \frac{\varepsilon H_0}{1-\varepsilon}(v-u)$$

where ρ_s and ρ_f are the densities of particles and gas, respectively; ε is the effective volume occupied by particles; u and v are the velocities of particle and gas, respectively; p is the pressure; G is the acceleration of gravity; Φ is the model force of phase interaction; a is the particle radius; ν is the kinematic viscosity of gas, and H_0